Cyclic inequality with three variables involving fractions.
https://www.linkedin.com/feed/update/urn:li:activity:6654693386425106432
Let a,b,c be nonnegative real numbers such that $a+b+c=4.$ Prove that
$$\frac{a^2b}{3a^2+b^2+4ca}+\frac{b^2c}{3b^2+c^2+4ab}+\frac{c^2a}{3c^2+a^2+4bc}\le\frac{1}{2}.$$
Solution by Arkady Alt, **San Jose**, **California**, **USA**.
We can assume for further that $a,b,c>0$ because if one of numbers a,b,c equal to zero (let it be $c=0$) then we obtain $\frac{a^2b}{3a^2+b^2}\le\frac{1}{2}\iff 3a^2+b^2-2a^2b\ge 0$
and $3a^2+b^2-2a^2b=3a^2+(4-a)^2-2a^2(4-a)=2(a+2)(a-2)^2\ge 0.$
In homogeneous form the inequality of the problems becomes
(**1**) $\sum\frac{a^2b}{3a^2+b^2+4ca}\le\frac{a+b+c}{8}$
Since $3a^2+b^2+4ca=2a^2+(a^2+b^2)+4ca\ge 2a^2+2ab+4ca=2a(a+b+2c)$
then $\sum\frac{a^2b}{3a^2+b^2+4ca}=\frac{a^2b}{2a^2+(a^2+b^2)+4ca}\le\frac{a^2b}{2a^2+2ab+4ca}=\frac{ab}{2(a+b+2c)}$
and, therefore, remains to prove inequality
(**2**) $\sum\frac{ab}{a+b+2c}\le\frac{a+b+c}{4}.$
Denoting $p:=ab+bc+ca,q:=abc$ and assuming $a+b+c=1$(due homogeneity of the inequality (2)) we obtain $\frac{a+b+c}{4}-\sum\frac{ab}{a+b+2c}=\frac{1}{4}-\sum\frac{ab}{1+c}=$
$\frac{1}{4}-\frac{\sum ab(1+a)(1+b)}{(1+a)(1+b)(1+c)}=\frac{1}{4}-\frac{p+p-3q+p^2-2q}{2+p+q}=\frac{2-7p-4p^2+21q}{4(2+p+q)}.$
Since $3p=3(ab+bc+ca)\le(a+b+c)^2=1$ and $q\ge q_*:=\max\left\{0,\frac{4-1}{9}\right\}$
because $9q\ge 4p-1$ (Schure's Inequality $\sum a(a-b)(a-c)\ge 0$ in p,q notation and normalized by $a+b+c=1$) we obtain $2-7p-4p^2+21q\ge 2-7p-4p^2+21q_*.$
For $p\in[1/4,1/3]$ we have $2-7p-4p^2+21q_*=2-7p-4p^2+21\bullet\frac{4p-1}{9}=$
$\frac{1}{3}(4p-1)(1-3p)\ge 0$ and for $p\in[0,1/4]$ we have $2-7p-4p^2+21q_*=$
$2-7p-4p^2=(p+2)(1-4p)\ge 0.$